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ON CORRECTNESS OF THE CAUCHY PROBLEM FOR A TWO-FLUID MODEL OF A GAS FLOW CONTAINING PARTICLES*

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A one-dimensional, nonstationary flow of a mixture of gas containing a dispersion of solid particles is used to investigate the correctness of the Cauchy problem within the framework of a two-fluid model /1/. The analysis is carried out with and without the volume occupied by the particles, taken into account. In both cases the norms are defined within which the problem is correct, even when the "fine ripples" appearing on the initial data cause the intersection of the particle trajectories upon which their volume density becomes infinite. The possibility of introducing the norms within which a problem, incorrect in some norm /2/, becomes correct without changing the model, is of major importance, since the correctness of the Cauchy problem, which does not represent another (straight) problem posed in different formulation, is considered as a natural requirement for the mathematical models of real processes /3,4/.

1. Using the two-fluid model approximation /1/ we describe the one-dimensional nonstationary flow of a mixture of gas and solid particles in the domains of continuity of the parameters, by the equations

$$\begin{gathered}
\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0,\quad \frac{Du}{Dt}+\frac{1}{\rho^\circ}\frac{\partial p}{\partial x}+\frac{\rho_s}{\rho}f=0 \\
\frac{\partial\rho_s}{\partial t}+\frac{\partial(\rho_s u_s)}{\partial x}=0,\quad \frac{D^s u_s}{Dt}+\mu\frac{\partial p}{\partial x}=f,\quad \frac{D\mathbf{v}_\tau}{Dt}+\frac{\rho_s}{\rho}\mathbf{f}_\tau=0 \\
T\frac{Ds}{Dt}=\frac{Di}{Dt}-\frac{1}{\rho^\circ}\frac{Dp}{Dt}=-\frac{\rho_s}{\rho}\sigma,\quad \frac{D^s\mathbf{v}_{s\tau}}{Dt}=\mathbf{f}_\tau,\quad \frac{D^s e_s}{Dt}=q \\
\rho=\rho^\circ-\nu\rho_s,\quad \rho^\circ=\rho^\circ(p,\ T),\quad i=i(p,\ T),\quad s=s(p,\ T) \\
e_s=e_s(T_s),\quad \mathbf{f}=\varphi_f\cdot(\mathbf{v}-\mathbf{v}_s),\quad q=\varphi_q\cdot(T-T_s) \\
\sigma=q-(\mathbf{v}-\mathbf{v}_s)\mathbf{f},\quad \mu=1/\rho_s^\circ,\quad \nu=\rho^\circ/\rho_s^\circ \\
(D/Dt=\partial/\partial t+u\partial/\partial x,\quad D^s/Dt=\partial/\partial t+u_s\partial/\partial x)
\end{gathered}\tag{1.1}$$

Here t is time, x is the space variable p is pressure, ρ° and ρ are the real and "blurred" densities, i and s are the specific enthalpy and entropy, T is absolute temperature, and $\mathbf{v}$ is the velocity of gas with the components u and $\mathbf{v}_\tau$ where u is the projection on the x-axis and $\mathbf{v}_\tau$ on the plane normal to the x-axis, the subscript s denotes the analogous parameters of the second phase (e_s is the specific internal energy), D/Dt and D^s/Dt are the total differentiation operators along the gas and the particles trajectories, $\mathbf{f}$ is the part of the force acting on the particles from the direction of the gas and governed by the difference in their velocities, f and $\mathbf{f}_\tau$ are the projections of $\mathbf{f}$ analogous to u and $\mathbf{v}_\tau$, q is the thermal flux from the gas towards the particles ($\mathbf{f}$ and q are referred to unit mass of the particles), φ_f and φ_q are known nonnegative functions of thermodynamic variables and $|\mathbf{v}-\mathbf{v}_s|$, $\rho^\circ(p,T),\ldots$ are known functions of their arguments. All parameters are assumed dimensionless and made so by means of the constants t_c, v_c, ρ_c and c_c which have dimensions of time, velocity, density and specific heat capacity. In addition $t_f=\varphi_f^{-1}$ and $t_q=\varphi_q^{-1}(\partial e_s/\partial T_s)$ assumes the sense of the dynamic and thermal relaxation times referred to t_c.

We formulate the Cauchy problem in question as follows. We define at $t=0$ the constant distributions of all parameters

$$\mathbf{v}(0,\ x)=\mathbf{v}_0,\ T(0,\ x)=T_0,\ \ \mathbf{v}_s(0,\ x)=\mathbf{v}_{s0},\ \ldots\ (-\infty<x<\infty)\tag{1.2}$$

with constants $\mathbf{v}_0\neq\mathbf{v}_{s0}$ and $T_0\neq T_{s0}$. Then for $t>0$ the parameters will, by virtue of (1.1) and (1.2), be independent of x, and ρ,ρ° and ρ_s also independent of t. Taking this into account, we can describe the flow using ordinary differential equations which can be obtained

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*Prikl.Matem.Mekhan.,46,No.3,pp.420-428,1982 by taking (1.1) without the first and third equation, deleting from it the derivatives in x, and replacing $\partial / \partial t$ by d / dt. The four combinations of these equations defining the projections of the impulse and energy of the mixture can be integrated, thus making it possible, in particular, to obtain at once the asymptotic (equilibrium) values of velocity ($\mathbf{v}_e$) and temperature (T_e) of the mixture

$$\mathbf{v}_s = \mathbf{v} = \mathbf{v}_e, \quad T_s = T = T_e \tag{1.3}$$

When $\mathbf{v}_{s0} \neq \mathbf{v}_0$ and $T_{s0} \neq T_0$, the relations (1.3) are strictly fulfilled only as $t \to \infty$. However, if $|\delta \mathbf{v}| \equiv |\mathbf{v}_s - \mathbf{v}| \ll |\mathbf{v}|$ and $|\delta T| \equiv |T_s - T| \ll T$, then further approach to the equilibrium is exponential $\delta \mathbf{v} = (\delta \mathbf{v})_0 \exp(-t / t_{fe})$ and $\delta T = (\delta T)_0 \exp(-t / t_{qe})$, and the equilibrium is established at $t \gg t_r$ where $t_r = \max(t_f, t_q)$.

The problem with initial conditions for a system of ordinary differential equations has, in the present case, a unique solution. For this reason the analysis of the correctness of the Cauchy problem (1.1) and (1.2) reduces to investigating, how the perturbation in the initial conditions (1.2) as dependent of x is reflected in the solution after any finite length of time. Taking this time as t_c, we consider the behavior of the perturbed solution at $0 < t \leqslant 1$. The initial perturbation of any parameter φ is specified in the form of "fine ripples", setting

$$\varphi(0, x) = \varphi_0 + \varphi_0' \equiv \varphi_0 + \varepsilon^\varphi \sin kx \quad (-\infty < x < \infty) \tag{1.4}$$

Here φ_0 denotes the unperturbed value of (1.2), ε^φ is the perturbation amplitude, k and $l = 2\pi / k$ are the wave number and wave length. Since the quantities in (1.4) are dimensionless, it follows that $\varphi_0 = O(1)$ and $\varepsilon^\varphi \ll 1$. The wave length is referred to $v_c t_c$ and k to the inverse quantity.

Depending on the initial differences in the phase velocities and temperatures, the perturbations in the values of the parameters u_0', u_{s0}', ... and the quantity t_r, the whole interval $0 < t \leqslant 1$ or its subintervals, are of the equilibrium (E) or nonequilibrium (N) type. We shall assign to the equilibrium (nonequilibrium) type the time intervals on which the deviations of the unperturbed flow parameters from their equilibrium values (1.3) are small (large). In turn, the intervals of the type N can be subdivided into the subsonic (SB) and supersonic (SP) subintervals in accordance with the characteristic properties of the system (1.1). The characteristic form in this system is assumed, before everything else, by two vector equations (each with two projections for $\mathbf{v}_\tau$ and $\mathbf{v}_{s\tau}$) and two scalar equations, i.e. six of the ten partial differential scalar equations, irrespective of the values of the parameters of the mixture. Three of these equations, namely the fifth (vector) and the sixth, are written along the gas trajectories, and the remaining three, i.e. the seventh (vector) and the eighth along the particle trajectories. The type of the subsystem of the first four equations of (1.1), connected with those listed above only by the coefficients and free terms, is determined by the number of real roots of the characteristic equation /5/

$$f(X, \Delta, \chi) \equiv (x - \Delta)^2 (x^2 - 1) - \chi x^2 = 0 \tag{1.5}$$
$$X = (u - x^{\cdot}) / a, \quad \Delta = (u - u_s) / a, \quad \chi = \rho_s \mu \nu \rho^\circ / \rho$$
$$a^{-2} = \rho_p^\circ + \rho_T^\circ (1 - \rho^\circ i_p) / (\rho i_T)$$

in which $x^{\cdot} \equiv dx / dt$ gives the slope of the characteristic in the xt-plane, a is the speed of sound in the gas and the subscripts p and T are assigned to the corresponding partial derivatives.

The compatibility conditions which hold on the characteristics of this subsystem, have the form /5/

$$u^{\cdot} - \frac{X}{\rho^\circ a} p^{\cdot} + X \frac{\rho_s \rho_T^\circ a}{\rho \rho^\circ i_T} \sigma + \frac{\rho_s}{\rho} f + \frac{\nu \rho_s X}{\rho (X - \Delta)} \left\{ \frac{X}{X - \Delta} (u_s^{\cdot} - f) + (u_s - u) \frac{\rho_s^{\cdot}}{\rho_s} \right\} = 0 \tag{1.6}$$

where, as in (1.5), a dot denotes the total derivative in t along the characteristic. If $\chi > 0$ and

$$|\Delta| > \Delta_* \equiv (1 + \chi^{1/3})^{3/2} \tag{1.7}$$

then, according to /1/, (1.5) has four different real roots $X_1, \ldots, X_4$, and the first four equations of (1.1) can therefore be replaced by the equivalent characteristic system obtained by making the substitution $X = X_j$ c $j = 1, \ldots, 4$ in (1.6). When $|\Delta| \leqslant \Delta_*$, such a substitution becomes impossible (if $\chi \neq 0$ and $|\Delta| < \Delta_*$, then only two equations can be replaced by the characteristic equations). Therefore, when $\chi \neq 0$, the time intervals on which the inequality (1.7) does (does not) hold, i.e. on which the difference between the x-components of the gas and particle velocities is supersonic (subsonic) with the accuracy of up to a small (in real situations) deviation of Δ_* from unity, shall be referred to the supersonic (subsonic) type. In the simplified model which neglects the volume of the particles, for which

μ, ν and χ are all equal to zero, the equation of continuity of the particles cannot be replaced by the equation of the characteristic type no matter when Δ are, although, when $\chi = 0$, (1.5) has four real roots ($X_{1,2} = \mp 1$ or $\dot{x}_{1,2} = u \pm a$ correspond to the $c^{\pm}$ characteristics and $X_{3,4} = \Delta$ or $\dot{x}_{3,4} = u_s$ to the particle trajectories). For this reason, the subdivision into the SB- and SP-intervals is not carried out in such model.

If the total time interval consists of various type intervals (e.g. SP, SB and E), then the analysis of the correctness of the whole problem reduces to consecutive investigations of several Cauchy problems, the first problem with the initial data (1.4) and the subsequent problems with initial data obtained from the solutions of the preceding problems (assuming that they are correct). In the course of considering each problem, its initial and final time is calculated conveniently by relating them to $t = 0$ and $t = 1$. Clearly, the correctness of the Cauchy problem for the total interval presupposes not only its correctness on its subintervals, but also the matching of the "initial" and "final" norms. The latter means that each preceding problem must be correct in that norm of the final results, which ensures the correctness of the following problems in which those results are used as the initial data.

The first stage of investigating the correctness is based on the linearization of (1.1) and the assumption that perturbations of all parameters are small. The solution obtained as a result of the linearization is sought in the form

$$\varphi'(t, x) = \operatorname{Im} \sum_j A_j^{\varphi}(k, t) \exp ik\{x - \vartheta_j(k, t)\} \tag{1.8}$$

where $\varphi'(t, x)$ denotes a perturbation of any parameter which can be represented in the form of a sum of φ' and the unperturbed solution $\varphi(t)$, j is the number of linearly independent solutions of the system, A_j^{φ} and ϑ_j are the corresponding complex amplitudes and phases. From (1.8) and (1.4) it follows that for all φ and j

$$\vartheta_j(k, 0) = 0, \qquad \operatorname{Im} \sum_j A_j^{\varphi}(k, 0) = \varepsilon^{\varphi}, \qquad \operatorname{Re} \sum_j A_j^{\varphi}(k, 0) = 0 \tag{1.9}$$

The right-hand sides of (1.8) are particularly simple for the E intervals on which the unperturbed parameters are constant, since in this case $\lambda_j \equiv \partial\vartheta_j / \partial t$ are constants and A_j^{φ} are either constants, or polynomials (for the multiple λ_j). We find however that for the N-intervals the variability of φ and hence of the coefficients of linear system does not cause any serious difficulties, since in this case only $k \to \infty$ are important. The latter is due to the boundedness of A_j^{φ} and ϑ_j at finite k and $0 < t \leqslant 1$ which occur, as a rule, in such problems. For this reason the incorrectness is usually not connected with the finite k, since in this case the smallness of $\varphi'(1, x)$ can always be secured by reducing the required number of times the amplitudes ε^{φ} of the initial perturbations which determine, according to (1.9), $A_j^{\varphi}(k, 0)$.

Limiting ourselves, by virtue of the above arguments, to large k, we shall continue, as in /3/, using when required the method of many scales /6/. Using this approach we establish, before anything else, that in the principal orders in $1/k$ the right-hand sides of (1.8) for v_s', $v_{s\tau}'$, s' and e_s' each contain a single term with amplitudes assuming finite values as $k \to \infty$, with $\lambda = u(t)$ for v_τ' and s' and $\lambda = u_s(t)$ for $v_{s\tau}'$ and e_s'. According to (1.9) and the definition of λ_j, we have

$$\vartheta_j(k, t) = \int_0^t \lambda_j(k, \tau)\, d\tau \tag{1.10}$$

and this implies that the terms of (1.8) with the amplitudes bounded as $k \to \infty$ and the real λ_j, and hence v_τ', $v_{s\tau}'$, s' and e_s', are all bounded. In contrast with this, the expressions (1.8) for u', p', u_s' and ρ_s' can contain up to four terms, and λ_j, which define according to (1.10) the corresponding phases, are roots of the "dispersion" equation (1.5) with $X = (u - \lambda) / a$. When $\chi \neq 0$, the equation has on the SB-intervals, in addition to two real roots $X_{1,2} = \mp 1 + O(\chi)$, two complex conjugate roots, which implies at the first glance the incorrectness of the Cauchy problem. In fact however the argument given here which holds for a specified norm, by no means excludes the possibility of introducing such norms in which the correctness is apparent on the intervals of all types. Before demonstrating such a possibility for (1.1), we turn our attention to the simplified model which disregards the volume ($\nu = \mu = \chi = 0$) of the particles. This enables us to clarify a number of fundamental aspects and to simplify the analysis of the case $\chi \neq 0$. Moreover, comparing the laws applicable to the models $\chi = 0$ and $\chi \neq 0$ we can explain the reasons behind the appearance of numerical instabilities which, although apparent in the computations with $\chi \neq 0$, do not materialize when a simplified model is used.

2. Let us consider the model $\chi = 0$. Remembering that its only interesting aspect is that of $k \to \infty$ we turn, before anything else, to the dispersion equation (1.5). As we have already said, at $\chi = 0$ all its roots are real ($X_{1,2} = \mp 1$, $X_{3,4} = \Delta$) and one of the first four equations of (1.1) with $\mu = \nu = 0$, i.e. the equation of continuity of the particles, cannot be reduced to the characteristic form. Therefore, taking into account (1.9) and (1.10) we find, that all ϑ_j are real, u', p' and u_s' are bounded, and

$$\rho_s' = \varepsilon^{\rho s} \sin k\zeta - \rho_s k t \varepsilon^{us} \cos k\zeta, \qquad \zeta = x - \int_0^t u_s(\tau)\, d\tau \approx x - u_{s0}t \tag{2.1}$$

The appearance in the solution of the terms proportional to kt may imply the incorrectness of neglecting the free terms in the linearized equation in the course of deriving the dispersion equation for $k \to \infty$. However, the analysis carried out for φ_f independent of ρ_s, with those terms taken into account, results only in a minor modification in the expressions for u', p' and u_s' which remained bounded as before, and in the appearance in (2.1) of additional (also bounded) terms, as well as change of the coefficients accompanying $\varepsilon^{\rho s}$ and ε^{us}. In particular, the term proportional to kt is replaced by a sum of terms of the form $k\varepsilon^{\varphi} f(k\zeta)\, \alpha^{-1} \exp \alpha t$ where $f(k\zeta) = \sin k\zeta$ or $\cos k\zeta$, and α are constants independent of k and connected with the coefficients of the linearized equations. It is for the latter reason that taking into account the free terms is not, in this case, of principal importance, since it does not affect the rate of growth of the perturbations in k as $k \to \infty$.

According to (2.1) and results of a more exhausting analysis taking into account the free terms of the equations, the smallness of ε^{φ} does not guarantee the smallness of ρ_s' when $t = 1$, since k can be arbitrarily large. Although this demonstrates the incorrectness of the Cauchy problem under the same perturbations norms for $t = 0$ and $t = 1$, the situation changes if we include in the set of initial perturbations not only the perturbations of the parameters themselves, but also of their first derivatives in x, $u'_{sx0} \sim k\varepsilon^{us}$ before anything else.

The possibility indicated here is not however unique. Indeed, the problem in question specifies unbounded (for fixed ε^{us} and $k \to \infty$) the growth of ρ_s', with the perturbations of the other parameters remaining within the order of their initial values. Keeping this in mind, we turn our attention to the integral law of conservation of particle mass which for an arbitrary closed contour Γ in the xt-plane, has the form

$$\oint_\Gamma (\rho_s\, dx - \rho_s u_s\, dt) = 0 \tag{2.2}$$

where ρ_s and u_s are understood to represent the complete (initial + perturbations) parameters. Applying (2.2) to the contour composed of the straight line segments $t = 0$ and $t = \text{const} \leqslant 1$ with their ends connected by the perturbed particle trajectories, and taking into account the smallness of u_s', we find that

$$m'(x_a, x_b, t) \equiv \int_{x_a}^{x_b} \rho_s'(t, \xi)\, d\xi = O(\rho_s \varepsilon^{us} t + \varepsilon^{\rho s} k^{-1}) \tag{2.3}$$

for any segment ab of the straight line $t = \text{const} \leqslant 1$.

It is important that the estimate (2.3) was obtained without making use of the periodicity of the solution in x, the latter causing for such segments $m' = O(\varepsilon^{\rho s} k^{-1})$ for any u_s'. It is for this reason only that the estimate in question which holds at small u_s' for any segments different from the particle trajectories, and for the nonperiodic solutions, is of a certain interest.

By virtue of (2.3) the Cauchy problem is correct if we replace in the set of the results ρ_s' by m', without including the derivatives in x in the set of initial data. Since a small change in m ensures a small change in the mean particles density $m / (x_b - x_a)$, it follows that such a replacement is natural and justified. Using as ab the segments of length small compared with the characteristic dimension (unity in this case), but finite, enables us to obtain, over m, the smoothed distributions ρ_s which are, in fact, necessary. This possibility is particularly interesting for a model with $\chi \neq 0$ where the incorporation of the derivatives into the set of initial data does not always make the incorrect problem correct. Keeping this in mind, we shall show that in the case of initial perturbations with unbounded derivatives ($k \to \infty$ at fixed $\varepsilon^{\varphi} \leqslant 1$) the nonlinear effect result in the intersection of the particle trajectories and, in contrast to (2.1), ρ_s becomes infinite at finite $t < 1$. We note that in /2/ such intersections are regarded as the cause of the incorrectness of the Cauchy problem.

At the points of intersection $\rho_s' \to \infty$, with u', p', ... of the order of u_0', p_0', ... Although the results of the linear analysis of (1.1) with $\mu = \nu = 0$, which confirm the assertion, may become invalid near these points, this in fact applies only to ρ_s'. The latter results from the structure of the equations for u, p, s and u_s which, for $\mu = \nu = 0$, are either the characteristic equations with ρ_s appearing only in the free terms, or can be converted to such. Integrating every characteristic equation from $t = 0$ to any point and using (2.2) for estimating the integrals in ρ_s' along the segments of the gas $c^{\pm}$- characteristics and trajectories, we obtain estimates of the form (2.3). If φ_f is independent of ρ_s, then the smallness of u', p', s' and u_s' at any ρ_s' follows at once, otherwise we find that the dependence of φ_f on ρ_s is usually so weak that it does not contribute, similarly as the dependence of φ_f on other parameters, any significant corrections to the formula determining the instant of intersection of the particle trajectories obtained for $\varphi_f = \text{const}$.

The instant of intersection of the particle trajectories is determined from the condition $x_\zeta \equiv (\partial x / \partial \zeta)_t = 0$ in which the new independent variable ξ, constant on each perturbed trajectory, varies on passage from one trajectory to the next. Taking, as in (2.1), ζ as the particle coordinate at $t = 0$, we find that

$$\zeta = x(t, \zeta) - \int_0^t u_s(\tau, \zeta)\, d\tau \tag{2.4}$$

where $u_s(t, \zeta)$ is the solution of the equation $u_{st} \equiv (\partial u_s / \partial t)_\zeta = \varphi_f \cdot (u - u_s)$. Here, unlike before, u and u_s represent total velocities and not their unperturbed values depending only on t. The formula for x_ζ is obtained by differentiating (2.4) and has the form

$$x_\zeta(t, \zeta) = 1 + \int_0^t u_{s\zeta}'(\tau, \zeta)\, d\tau \equiv 1 + \int_0^t u_{s\zeta}(\tau, \zeta)\, d\tau$$

Let us integrate (at $\varphi_f = \text{const}$, which is in fact unimportant), the equation for u_s, differentiate the result obtained with respect to ζ and substitute $u_{s\zeta}$ into the previous formula. This yields

$$\begin{gathered} x_\zeta = 1 + (1 - e^{-t/t_f})\, t_f u_{sx0}' + \frac{1}{t_f}\int_0^t e^{-\tau/t_f}\, d\tau \int_0^\tau u_\zeta'(\xi, \zeta)\, e^{\xi/t_f} d\xi = 1 + \\ k\varepsilon^{us}(1 - e^{-t/t_f})\, t_f \cos kx + t^2 \cdot O(\varepsilon^u + \varepsilon^p + k\varepsilon^{us} t + \varepsilon^{\rho s}) \end{gathered} \tag{2.5}$$

In carrying out the above manipulations we have made use of the initial conditions (1.4) and estimates given above. Moreover, we have taken into account the alternating signs of the contributions to u_ζ' furnished by the perturbations arriving along the gas $c^{\pm}$-characteristics and trajectories. According to the equation for particle continuity $\rho_s + \rho_s' = (\rho_s + \rho_{s0}') / x_\zeta$, where $\rho_s \equiv \rho_{s0}$. Since $t \leqslant 1$, this, together with (2.5), yield for $k\varepsilon^{us} \ll 1$ the formula (2.1) in the linear approximation. Conversely, for $k\varepsilon^{us} \geqslant O(1)$ the density of the particles may become infinite already at finite t. For any fixed $\varepsilon^{us} \ll 1$ and $k \to \infty$ this takes place at $t = t_* = 1/(k\varepsilon^{us})$.

The intersection of the particle trajectories under the initial conditions (1.4) leads, within the framework of the two-liquid model in question, to formation of a set of discontinuity surfaces in the form of sheets (sheaves or clusters) /1,7,8/, with finite surface density R_s. The particles which occupied at $t = 0$ the segments of the x-axis of length $l = 2\pi / k$ arrive at each sheet over the time interval of the order of t_*, counted from its appearance. This makes R_s equal to $2\pi\rho_{s0} / k$. However, even in such a situation the conclusions made about whether the problem is ocrrect or incorrect depend on the type of perturbations experienced by the other parameters. In particular, if the velocity of the sheet U_s differs at $t \leqslant 1$ little from the velocity of the particles in the unperturbed solution, then we can pass again, as before, in the set of the results, from ρ_s' to m', use the estimate (2.3) and obtain, if necessary, in terms of m, the smoothing of the particle density without taking into account their "bunching".

In the course of forming the particle agglomerates over the period of order k^{-1}, their parameters different from R_s will in fact coincide with u_s, $\mathbf{v}_{s\tau}$ and e_s by virtue of the conditions of conservation of particle mass, impulse and energy, since the interactions between the particles and gas over such short time intervals result in contributions of the order of k^{-1}. Further variation in the parameters of the sheet on the N-intervals occurs when it reacts with the gas without depositing new particles ($R_s = \text{const}$), and is described by the equations /1/

$$dU_s / dt = F / R_s,\ d\mathbf{V}_{s\tau} / dt = \mathbf{F}_\tau / R_s,\ dE_s / dt = Q \tag{2.6}$$

in which d/dt denotes the total derivative along the trajectory of the sheet, $dx / dt = U_s$, $\mathbf{V}_{s\tau}$ is the tangential sheet velocity component, E_s is its specific internal energy, F and $\mathbf{F}_\tau$ are the components of the surface force $\mathbf{F}$ acting from the gas on a unit surface of the sheet, and

Q is the amount of heat passing from the gas to unit mass of the sheet in unit time.

Let the gas parameters in the perturbed and unperturbed flow be almost equal. Then comparing (2.6) with (1.1) we see that U_s, $\mathbf{V}_{s\tau}$ and E_s will be nearly equal to u_s, $\mathbf{v}_{s\tau}$ and e_s of the unperturbed flow provided that

$$F / R_s \to f, \quad \mathbf{F}_\tau / R_s \to \mathbf{f}_\tau, \quad Q \to q \quad \text{for} \quad R_s \equiv 2\pi\rho_{s0} / k \to 0 \tag{2.7}$$

where f, $\mathbf{f}_\tau$ and q are the same as in (1.1) but after replacing the particle parameters with the sheet parameters. We can stipulate that the conditions (2.7) essential for any model of the sheet, ensure also the smallness of the perturbations in gas parameters. Indeed, integrating in the unperturbed flow the conditions of compatibility (1.6) with $\nu=0$ and the equations for s and $\mathbf{v}_\tau$ from (1.1) across the strip of width $l = 2\gamma/k$ bounded by the particle trajectories, we find with the accuracy of $O(k^{-2})$, that

$$\Delta u \pm \frac{\Delta p}{\rho^\circ a} = \frac{\rho_s l a^\pm}{u - u_s \pm a}, \quad \Delta s = \frac{\rho_s l}{u_s - u} b, \quad \Delta \mathbf{v}_\tau = \frac{\rho_s l \mathbf{f}_\tau}{\rho (u - u_s)} \tag{2.8}$$
$$\left(a^\pm = \frac{\rho_T{}^\circ a \sigma \mp \rho^\circ i_T f}{\rho^{\circ 2} i_T}, \quad b = \frac{\sigma}{\rho T} \right)$$

where $\Delta\varphi$ is the increment in φ during the passage across the strip in direction of increasing t, and the $+$ $(-)$ signs correspond to the c^+ (c^-) segments of the characteristics. In the perturbed motion, after the particles of this strip have been deposited on the sheet, the left-hand sides of (2.8) vary only because of the interaction between the gas and the sheet, in accordance with the conditions /1/

$$[\rho (u - U_s)] = 0, \quad [p + \rho (u - U_s)^2] = -F, \quad [\rho (u - U_s)\mathbf{v}_\tau] = -\mathbf{F}_\tau \tag{2.9}$$
$$[\rho (u - U_s) \{2i + (u - U_s)^2 + v_\tau^2\}] = -2 (R_s Q + \mathbf{F}_\tau \mathbf{V}_{s\tau})$$

in which $[\varphi] = \varphi^+ - \varphi^-$ and φ^- (φ^+) denote the value of φ in front (behind) the sheet in the direction of the gas flow. By virtue of (2.9) we have, with the accuracy of up to $O(R_s^2)$,

$$[u] \pm \frac{[p]}{\rho^\circ a} = \frac{R_s A^\pm}{u - U_s \pm a}, \quad [s] = \frac{R_s B}{U_s - u}, \quad [\mathbf{v}_\tau] = \frac{\mathbf{F}_\tau}{\rho (u - U_s)} \tag{2.10}$$
$$\left(A^\pm = \frac{\rho_T{}^\circ a \Sigma \mp \rho^\circ i_T F / R_s}{\rho^{\circ 2} i_T}, \quad B = \frac{\Sigma}{\rho T}, \quad \Sigma = Q - (\mathbf{v} - \mathbf{V}_s) \frac{\mathbf{F}}{R_s} \right)$$

Here $\mathbf{V}_s = \mathbf{n} U_s + \mathbf{V}_{s\tau}$, $\mathbf{n}$ is the unit vector in the x-direction and, in contrast to (2.9), the parameter differences are brought in analogously to $\Delta\varphi$ in (2.8), in the direction of increasing t, along the corresponding line. Since $R_s = \rho_s l$, comparing (2.8) with (2.10) proves the assertion made above.

Since every sheet is described by the ordinary differential equations (2.6) with $R_s =$ const, it follows, that for a model disregarding the volume of the particles, the Cauchy problem on the N-intervals is correct for the set of initial data without the derivatives in x, and the set of results with m' replacing ρ_s'. An analogous assertion for the E-intervals is proved more simply, since on these intervals the properties of the gas and particle agglomerates after forming into sheets remain almost unchanged. Moreover, here and in Sect.3 no assumption is required concerning the dependence of φ_f on the parameters.

3. Returning now to the model with $\chi \neq 0$, we first write down the results of linear analysis commenced in Sect.1. On the SP-intervals where all roots of the dispersion equation (1.5) are real, the Cauchy problem is correct under the usual conditions, at least when the perturbations do not affect the validity of the linear analysis. Perusal of the possible intersections of the particle trajectories shows that the latter means the boundedness of the initial derivatives $\varphi_{x0}' \sim k\varepsilon^\varphi < K^\varphi$ where $K^\varphi = O(1)$ are constants depending from unperturbed solution. For the E-intervals the results of the linear analysis of the models with $\chi \neq 0$ and $\chi = 0$, are identical. Thus the only increasing (linearly in k) parameter is ρ_s'. Therefore, by neglecting the free terms in the linearized equations and discarting from the results the quantities of the order of μ and ν, we can retain for ρ_s' the formula (2.1). Therefore the situation in the complete model with $\chi \neq 0$ does not differ on the E-interval from the situation realized for the simplified model on any intervals. The Cauchy problem is correct if the derivatives in x are included in the set of initial data.

The main difference between the models with $\chi \neq 0$ and $\chi = 0$ manifests itself on the SB-intervals which, incidentally, are the most interesting ones. Here one of the two complex conjugate roots of (1.5) with $\chi \neq 0$ yields an exponentially growing solution with the index proportional to k. Retaining only this root in (1.8) and the terms principal in μ and ν in the expression accompanying the increasing exponent, we obtain for $\chi \ll 1$

$$p' = \frac{\alpha\varepsilon^{us}}{2\mu} e^{\alpha kt} \cos k\zeta, \quad u' = -\frac{p'}{\rho^\circ a \Delta}, \quad u_s' = \frac{1}{2} \varepsilon^{us} e^{\alpha kt} \sin k\zeta, \quad \rho_s' = \frac{1 - \Delta^2}{\nu (a\Delta)^2} p' \tag{3.1}$$

$$\left(\alpha = \frac{a\Delta\sqrt{\chi}}{\sqrt{1-\Delta^2}} + o(\sqrt{\chi}),\ \ \zeta = x - \int_0^t \beta(\tau)\,d\tau, \quad \beta = u_s - \frac{a\Delta\chi}{1-\Delta^2} + o(\chi)\right)$$

The presence of k in the exponents prevents us from ensuring the correctness by the fact that arbitrary derivatives in x are included in the set of initial data, and this represents the main difference between the complete and the simplified model. Thus when $\chi \neq 0$ and the initial perturbations are arbitrarily smooth (in the functions themselves, as well as in their first, second, etc. derivatives), then by virtue of (3.1) the perturbations of the parameters, primarily ρ_s', can grow as strongly as we like, at $t > 0$. However, within the framework of the nonlinear analysis this takes plane only for ρ_s', and $\rho_s' = \infty$ at finite $t = t_*$, when the particle trajectories intersect for the first time. Thus, after the time of the order of t_*, as well as for $\chi = 0$, all particles coalesce and form agglomerates with surface density of $R_s = 2\pi\rho_{s0} / k$. The nonlinear analysis includes the same stages as the case $\chi = 0$, and leads to analogous results and conclusions. First, using (1.6), (2.2) and the estimates of the type (2.3), we show that for $t \leqslant t_*$ u' and p' are much smaller than ρ_s' and satisfy the formulas (3.1) with ζ constant on the perturbed particle trajectories. This enables us to find x_ζ and t_*. Thus for $k\varepsilon^{us} = \varepsilon$ with fixed $\varepsilon \ll 1$, i.e. with u_{sx0}' bounded, (2.5) is replaced by

$$x_\zeta \approx 1 + \varepsilon^{us} (2\alpha)^{-1} e^{\alpha k t} \cos k\zeta = 1 + \varepsilon\,(2\alpha k)^{-1} e^{\alpha k t} \cos k\zeta$$

The above formula, valid for $\alpha k \gg 1$ yields $t_* = (\alpha k)^{-1} \ln (2\alpha k / \varepsilon)$. For the nonsmooth initial perturbations t_* will be much smaller, in particular, if $k \to \infty$ and ε^φ is fixed, then, as in Sect.2, $t_* = 1 / (k\varepsilon^{us})$.

Further arguments identical for $\chi \neq 0$ and $\chi = 0$ include (2.6) – (2.10). Here (2.8) follows from (1.6) and at $\nu \neq 0$, just as in the unperturbed solution, $u^\cdot = f$ and $\rho_s^\cdot = 0$, and the validity of (2.9) and (2.10) is justified by the fact that when the process of coalesence of the particles is completed, no particles exist outside the agglomerates. Thus the Cauchy problem is also correct for $\chi \neq 0$ with m' replacing ρ_s' in the results, provided that conditions (2.7) are fulfilled. Since in the simplest model of the sheet /7/ the conditions hold for any R_s, therefore (2.7) is reduced to the requirement that any, more complete model, transforms to the simpler model as $R_s \to 0$.

The completed analysis can be applied without any difficulty to the initial distributions different from (1.2) and to other multiphase liquid models without inherent pressure of the dispersed phases. Although the individual details may change (e.g. in the case of two incompressible fluids with $\mu = 0$ and $\nu \neq 0$ studied in /2/, the exponential index in the analog of (3.1) is proportional, as $k \to \infty$, not to k but to $\sqrt{k}$), the conclusions about the correctness of the Cauchy problem remain valid. This justifies the regularization procedures smoothing the fine ripples which must, nevertheless, allow the formation of the macroaggregates.

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